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Linear-Nonquadratic Optimal Control Problems with Terminal Inequality Constraints

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This paper considers a linear-nonquadratic optimal control problem subject to nonlinear terminal inequality constraints. We approximate it by a series of approximate problems via the penalty method. It is shown that the optimal control functions of the approximate problems uniformly converge to the optimal control function of the original terminal constrained problem. Furthermore, the optimal values of the approximate problems also converge to the optimal value of the original problem under some mild conditions. © 1997 Academic Press

1. INTRODUCTION

Linear-quadratic optimal control problems with terminal constraints have been extensively investigated in the literature (for example, see [2–8] and the references cited therein). Linear-nonquadratic optimal control problems without terminal constraints have also been thoroughly studied (see [9–11]). Then, a natural question to ask is how to solve linear-nonquadratic optimal control problems with terminal constraints. In [8], a linear-nonquadratic optimal control problem with an inequality terminal constraint is handled by the maximum principle [1], leading to a two-point boundary-value problem. For the existence of optimal controls, some

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hypotheses such as the convexity assumption are made. In this paper, we consider a class of linear-nonquadratic optimal control problems. In Section 2, we formulate our optimal control problem, where the controlled system is a time-varying system with forcing term, the terminal constraints are described by nonlinear inequalities, and the cost functional is a nonquadratic Bolza type functional. Note that the terminal inequality constraints include those cases in which the constraint sets are described by linear manifolds or hyper-ellipsoids, or their finite intersections as special cases. In Section 4 and Section 5, we use the idea of the penalty method to handle our terminal constrained optimal control problem. We construct a sequence of linear-nonquadratic optimal control problems without terminal constraints. Under a controllability condition and appropriate convexity assumptions, we show that the optimal control functions of the approximate problems uniformly converge to the optimal control function of the original optimal control problem, while the optimal values of the approximate problems also converge to the optimal value of the original problem as the penalty constant tends to infinity.

2. PROBLEM STATEMENT

Let $C(S_1, S_2)$ (respectively, $C^1(S_1, S_2)$) denote the set of all continuous (respectively, continuously differentiable) functions defined in a subset S_1 of an Euclidean space with values in a subset S_2 of another Euclidean space. A measurable function $\mathbf{u}:[t_0, t_1] \to R^m$ is called an admissible control if it is such that

$$\int_{t_0}^{t_1} |\mathbf{u}(t)|^2 dt < +\infty,$$

where $|\cdot|$ denotes the usual Euclidean norm. Let U be the class of all admissible controls.

Now, we consider the following class of time-varying system involving a forcing term

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) + \mathbf{f}(t), \qquad t \in [t_0, t_1], \tag{2.1a}$$

$$\mathbf{x}(t_0) = \mathbf{x}^0, \tag{2.1b}$$

where $\mathbf{x} \in R^n$ and $\mathbf{u} \in R^m$ are the state and control vector, respectively, and $\mathbf{x}^0$ is a given vector in R^n. It is assumed that $A \in C([t_0, t_1], R^{n\times n})$, $B \in C([t_0, t_1], R^{n\times m})$, and $\mathbf{f} \in C([t_0, t_1], R^n)$.

For each $\mathbf{u} \in \mathcal{U}$, let $\mathbf{x}(\cdot \mid \mathbf{u})$ denote the corresponding solution of the system (2.1a)–(2.1b). Let $\mathcal{F}$ be a subset of $\mathcal{U}$ such that the following terminal inequality constraints are satisfied:

$$E_j(\mathbf{x}(t_1 \mid \mathbf{u})) \leq 0, \qquad j = 1, 2, \ldots, l. \tag{2.1c}$$

Elements of $\mathcal{F}$ are referred to as feasible controls and $\mathcal{F}$ is called the class of feasible controls. The optimal control problem considered in this paper may now be stated formally as follows.

Subject to dynamical system (2.1), find a feasible control $\mathbf{u} \in \mathcal{F}$ such that the cost functional

$$J(\mathbf{u}) = W(\mathbf{x}(t_1 \mid \mathbf{u})) + \int_{t_0}^{t_1} \left[Q(t, \mathbf{x}(t \mid \mathbf{u})) + \frac{1}{2} \mathbf{u}(t)^T R(t) \mathbf{u}(t) \right] dt \tag{2.2}$$

is minimized over $\mathcal{F}$, where $W \in C^1(R^n, R^1)$, $Q \in C([t_0, t_1] \times R^n, R^1)$, $Q(t, \cdot): R^n \to R^n \in C^1(R^n, R^1)$ for each $t \in [t_0, t_1]$, $R \in C([t_0, t_1], R^{m \times m})$ and the superscript T denotes transpose. Furthermore, it is assumed that $W(\cdot): R^n \to R^1$ is convex, $Q(t, \cdot): R^n \to R^1$ is convex, and there exists a $\delta_0 > 0$ such that

$$R(t) \geq \delta_0 I_m, \qquad \forall t \in [t_0, t_1], \tag{2.3}$$

where I_m denotes the identity matrix in $R^{m \times m}$.

Let this above optimal control problem be referred to as Problem (P).

3. EXISTENCE AND UNIQUENESS OF OPTIMAL CONTROLS

In this section, we shall establish results on the existence and uniqueness of optimal controls.

THEOREM 3.1. *If $\mathcal{F} \neq \emptyset$, then Problem* (P) *admits a unique optimal control.*

Proof. In view of the convexity assumptions on $W(\cdot): R^n \to R^1$ and $Q(T, \cdot): R^n \to R^1$ for each $t \in [t_0, t_1]$, we have

$$W(\mathbf{x}) \geq W(\mathbf{x}(t_1 \mid \mathbf{0})) + \left(\left. \frac{\partial W(\mathbf{y})}{\partial \mathbf{y}} \right|_{\mathbf{y} = \mathbf{x}(t_1 \mid \mathbf{0})} \right)^T (\mathbf{x} - \mathbf{x}(t_1 \mid \mathbf{0})), \qquad \forall \mathbf{x} \in R^n,$$

$$Q(t, \mathbf{x}) \geq Q(t, \mathbf{x}(t \mid \mathbf{0})) + \left(\left. \frac{\partial Q(t, \mathbf{y})}{\beta \mathbf{y}} \right|_{\mathbf{y} = \mathbf{x}(t \mid \mathbf{0})} \right)^T (\mathbf{x} - \mathbf{x}(t \mid \mathbf{0})),$$

$$\forall \mathbf{x} \in R^n, t \in [t_0, t_1]. \tag{3.1}$$

Let $G(\cdot, s):[t_0, t_1] \to R^n$ be the matrix solution of the homogeneous linear matrix differential system

$$\begin{cases} \dot{X}(t,s) = A(t)X(t,s), & (t,s) \in [s,t_1] \times [t_0,t_1], \\ X(s,s) = I_n \end{cases} \tag{3.2}$$

for any $s \in [t_0, t_1]$, where I_n denotes the identity matrix in $R^{n\times n}$. By (3.1) and (2.3), it follows from (2.2) that

$$\begin{aligned} J(\mathbf{u}) &\geq J(\mathbf{0}) + \left(\frac{\partial W(\mathbf{y})}{\partial \mathbf{y}}\bigg|_{\mathbf{y}=\mathbf{x}(t_1|\mathbf{0})} \right)^T \int_{t_0}^{t_1} G(t_1,\tau)B(\tau)\mathbf{u}(\tau)\,d\tau \\ &\quad + \int_{t_0}^{t_1} \left[\left(\frac{\partial Q(t,\mathbf{y})}{\partial \mathbf{y}}\bigg|_{\mathbf{y}=\mathbf{x}(t|\mathbf{0})} \right)^T \int_{t_0}^{t} G(t,\tau)B(\tau)\mathbf{u}(\tau)\,d\tau + \frac{\delta_0}{2}|\mathbf{u}(t)|^2 \right] dt \\ &\geq J(\mathbf{0}) + \int_{t_0}^{t_1} \left\{ \frac{\delta_0}{2}|\mathbf{u}(t)|^2 + \left[G(t_1,t)^T \frac{\partial W(\mathbf{y})}{\partial \mathbf{y}}\bigg|_{\mathbf{y}=\mathbf{x}(t_1|\mathbf{0})} \right.\right. \\ &\quad \left.\left. + \int_{t}^{t_1} G(\tau,t)^T \frac{\partial Q(\tau,\mathbf{y})}{\partial \mathbf{y}}\bigg|_{\mathbf{y}=\mathbf{x}(\tau|\mathbf{0})} d\tau \right]^T B(t)\mathbf{u}(t) \right\} dt \\ &\geq J(\mathbf{0}) + \int_{t_0}^{t_1} \left[\frac{\delta_0}{2}|\mathbf{u}(t)|^2 - C|\mathbf{u}(t)| \right] dt \\ &= J(\mathbf{0}) + \int_{t_0}^{t_1} \frac{\delta_0}{2} \left[|\mathbf{u}(t)| - \frac{C}{\delta_0} \right]^2 dt - \frac{C^2(t_1-t_0)}{2\delta_0} \\ &\geq J(\mathbf{0}) - \frac{C^2(t_1-t_0)}{2\delta_0}, \qquad \forall \mathbf{u} \in U, \end{aligned} \tag{3.3}$$

where

$$C := \max_{t_0 \leq t \leq t_1} \left| B(t)^T \left[G(t_1,t)^T \frac{\partial W(\mathbf{y})}{\partial \mathbf{y}}\bigg|_{\mathbf{y}=\mathbf{x}(t_1|\mathbf{0})} + \int_{t}^{t_1} G(\tau,t)^T \frac{\partial Q(\tau,\mathbf{y})}{\partial \mathbf{y}}\bigg|_{\mathbf{y}=\mathbf{x}(\tau|\mathbf{0})} d\tau \right] \right|.$$

Thus,

$$\inf_{\mathbf{u}\in F} J(\mathbf{u}) > -\infty. \tag{3.4}$$

On the other hand, since $F \neq \bigcirc$, we have

$$\inf_{\mathbf{u} \in F} J(\mathbf{u}) < +\infty. \tag{3.5}$$

Combining (3.4) and (3.5), $\inf_{\mathbf{u} \in F} J(\mathbf{u})$ is a finite real number. Clearly, there exists a sequence $\{\mathbf{u}_\mu\} \subseteq F$ such that

$$\lim_{\mu \to \infty} J(\mathbf{u}_\mu) = \inf_{\mathbf{u} \in F} J(\mathbf{u}). \tag{3.6}$$

From (3.3), it follows that

$$\begin{aligned} J(\mathbf{u}_\mu) \geq \int_{t_0}^{t_1} \frac{\delta_0}{2} \left[|\mathbf{u}_\mu(t)| - \frac{C}{\delta_0} \right]^2 dt \\ + J(\mathbf{0}) - \frac{C^2(t_1 - t_0)}{2\delta_0}, \qquad \mu = 1, 2, \ldots \end{aligned} \tag{3.7}$$

which yields

$$\begin{aligned} \int_{t_0}^{t_1} |\mathbf{u}_\mu(t)|^2 dt &\leq 2 \left\{ \int_{t_0}^{t_1} \left[|\mathbf{u}_\mu(t)| - \frac{C}{\delta_0} \right]^2 dt + \left(\frac{C}{\delta_0} \right)^2 (t_1 - t_0) \right\} \\ &\leq 4 \left\{ \frac{1}{\delta_0} \left[J(\mathbf{u}_\mu) - J(\mathbf{0}) \right] + \left(\frac{C}{\delta_0} \right)^2 (t_1 - t_0) \right\} \\ &\leq 4 \left\{ \frac{1}{\delta_0} \left[\sup_\mu J(\mathbf{u}_\mu) - J(\mathbf{0}) \right] + \left(\frac{C}{\delta_0} \right)^2 (t_1 - t_0) \right\} < +\infty, \\ &\qquad \mu = 1, 2, \ldots \end{aligned} \tag{3.8}$$

This, in turn, implies that $\{\mathbf{u}_\mu\}$ is a bounded sequence in $L_2(t_0, t_1; R^m)$ which is the Hilbert space consisting of all elements of U and the inner product defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle := \int_{t_0}^{t_1} \mathbf{v}(t)^T \mathbf{w}(t)\, dt, \qquad \forall \mathbf{v}, \mathbf{u} \in U.$$

Hence, there exist a subsequence $\{\mathbf{u}_{\mu_\nu}\}$ and a $\mathbf{u}_\infty \in U$ such that

$$\mathbf{u}_{\mu_\nu} \to \mathbf{u}_\infty \tag{3.9}$$

in the weak topology of $L_2(t_0, t_1; R^m)$ as $\nu \to \infty$. Then by the Banach–Saks theorem, there exist two positive integer sequences $\{\nu_p'\}$, $\{\nu_p''\}$, and a vector

$\{(\lambda_{\nu_{p'}}, \lambda_{\nu_{p'}+1}, \ldots, \lambda_{\nu''_p})^T\}$ such that

$$\nu'_1 < \nu''_1 < \nu'_2 < \nu''_2 < \ldots < \nu'_p < \nu''_p < \ldots, \tag{3.10}$$

$$\lambda_{\nu'_p} \geq 0, \lambda_{\nu_{p'}+1} \geq 0, \ldots, \lambda_{\nu_{p''}} \geq 0, \qquad \sum_{\nu=\nu_{p'}}^{\nu_{p''}} \lambda_\nu = 1, p = 1, 2, \ldots \tag{3.11}$$

and

$$\lim_{p\to\infty} \int_{t_0}^{t_1} \left| \sum_{\nu=\nu_{p'}}^{\nu_{p''}} \lambda_\nu \mathbf{u}_{\mu_\nu}(t) - \mathbf{u}_\infty(t) \right|^2 dt = 0. \tag{3.12}$$

Now, noting that $J(\cdot): L_2([t_0, t_1], R^m) \to R^1$ is convex and continuous, we have

$$\begin{aligned} J(\mathbf{u}_\infty) &= \lim_{p\to\infty} J\left(\sum_{\nu=\nu_{p'}}^{\nu_{p''}} \lambda_\nu \mathbf{u}_{\mu_\nu} \right) \leq \lim_{p\to\infty} \sum_{\nu=\nu_{p'}}^{\nu_{p''}} \lambda_\nu J(\mathbf{u}_{\mu_\nu}) \\ &= \lim_{\mu\to\infty} J(\mathbf{u}_\mu) = \inf_{\mathbf{u}\in F} J(\mathbf{u}) \end{aligned} \tag{3.13}$$

and

$$E_j\big(\mathbf{x}(t_1 \mid \mathbf{u}_\infty)\big) = \lim_{\mu\to\infty} E_j\big(\mathbf{x}(t_1 \mid \mathbf{u}_\mu)\big) \leq 0, \qquad j = 1, 2, \ldots, l. \tag{3.14}$$

Therefore,

$$\mathbf{u}_\infty \in F, \qquad J(\mathbf{u}_\infty) = \inf_{\mathbf{u}\in F} J(\mathbf{u}), \tag{3.15}$$

and hence, $\mathbf{u}_\infty$ is an optimal control to Problem (P). The uniqueness of the optimal controls follows readily from the strict convexity of the cost functional and the convexity of the feasible set F.

DEFINITION 3.1. We say that (A, B) is controllable on $[t_0, t_1]$ if the matrix

$$\int_{t_0}^{t_1} G(t_1, t) B(t) B(t)^T G(t_1, t)^T dt$$

is positive definite.

It is well known (see Lemma 1 in Section 13 of [1]) that for any $\mathbf{x}^1 \in R^n$, if (A, B) is controllable on $[t_0, t_1]$ then there exists a $\mathbf{u} \in U$ such that

$$\mathbf{x}(t_1 \mid \mathbf{u}) = \mathbf{x}^1. \tag{3.16}$$

Thus, as a direct consequence of Theorem 3.1, we have

COROLLARY 3.1. *Let (A, B) be controllable on $[t_0, t_1]$. If*

$$\left\{\mathbf{x} \in R^n \mid E_j(\mathbf{x}) \leq 0, j = 1, 2, \dots, l\right\} \neq \emptyset, \tag{3.17}$$

then Problem (P) admits a unique optimal control.

4. PENALTY METHOD

Using the concept of a penalty function, we can append the terminal constraints to the cost functional, leading to an augmented cost functional

$$J_\gamma(\mathbf{u}) := \gamma \sum_{j=1}^{l} g\big(E_j(\mathbf{x}(t_1 \mid \mathbf{u}))\big) + W(\mathbf{x}(t_1 \mid \mathbf{u})) + \int_{t_0}^{t_1} \left[Q(t, \mathbf{x}(t \mid \mathbf{u})) + \frac{1}{2}\mathbf{u}(t)^T R(t)\mathbf{u}(t)\right] dt, \tag{4.1}$$

where $\mathbf{u} \in \mathcal{U}$,

$$g(r) = \begin{cases} \dfrac{1}{2k} r^{2k}, & \text{if } r \geq 0, \\ 0, & \text{otherwise}, \end{cases}$$

k and γ are two positive integers. We now obtain an unconstrained optimal control problem for each positive integer γ, where the augmented cost functional $J_\gamma(\cdot)$ is to be minimized over $\mathcal{U}$ subject to system (2.1a) with initial condition (2.1b). Let this unconstrained optimal control problem be referred to as Problem (P_γ).

Note that $\sum_{j=1}^{l} g(E_j(\cdot)) + W(\cdot) \in C^1(R^n, R^1)$ and is convex. Furthermore, there is no terminal constraint involved in Problem (P_γ). Thus, as a special case of Theorem 3.1, we have

COROLLARY 4.1. *Problem (P_γ) admits a unique optimal control for each $\gamma > 0$.*

5. MAIN CONVERGENCE RESULT

Now, we present our main convergence result in the following theorem.

THEOREM 5.1. *Let (A, B) be controllable on $[t_0, t_1]$. If (3.17) is satisfied, then*

$$\lim_{\gamma \to \infty} \max_{t_0 \leq t \leq t_1} \left|\hat{\mathbf{u}}_\gamma(t) - \hat{\mathbf{u}}(t)\right| = 0, \tag{5.1}$$

where $\hat{\mathbf{u}}_\gamma$ *and* $\hat{\mathbf{u}}$ *are the optimal controls of Problem* (P_γ) *and Problem* (P), *respectively*. *Furthermore*,

$$\lim_{\gamma \to \infty} J_\gamma(\hat{\mathbf{u}}_\gamma) = J(\hat{\mathbf{u}}). \tag{5.2}$$

Proof. From Corollary 3.1 and Corollary 4.1, there exist $\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}_\gamma$ which are the unique optimal controls of Problem (P) and Problem (P_γ), respectively. By the maximum principle, there exists a $\hat{\psi}_\gamma \in C^1([t_0, t_1], R^n)$ such that

$$\hat{\mathbf{u}}_\gamma(t) = R(t)^{-1} B(t)^T \hat{\psi}_\gamma(t), \qquad \forall t \in [t_0, t_1] \tag{5.3}$$

and

$$\begin{cases} \dot{\hat{\psi}}_\gamma(t) + A(t)^T \hat{\psi}_\gamma(t) = \left.\dfrac{\partial Q(t, \mathbf{y})}{\partial \mathbf{y}}\right|_{\mathbf{y} = \mathbf{x}(t \mid \hat{\mathbf{u}}_\gamma)}, \qquad t \in [t_0, t_1], \\ \hat{\psi}_\gamma(t_1) = -\left[\gamma \displaystyle\sum_{j=1}^{l} h\Big(E_j\big(\mathbf{x}(t_1 \mid \hat{\mathbf{u}}_\gamma)\big)\Big) \left.\dfrac{\partial E_j(\mathbf{y})}{\partial \mathbf{y}}\right|_{\mathbf{y} = \mathbf{x}(t_1 \mid \hat{\mathbf{u}}_\gamma)} \right. \\ \qquad\qquad \left. + \left.\dfrac{\partial W(\mathbf{y})}{\partial \mathbf{y}}\right|_{\mathbf{y} = \mathbf{x}(t_1 \mid \hat{\mathbf{u}}_\gamma)} \right], \end{cases} \tag{5.4}$$

where

$$h(r) = \begin{cases} r^{2k-1}, & \text{if } r \geq 0, \\ 0, & \text{otherwise}. \end{cases} \tag{5.5}$$

Clearly, the first equality in (5.4) is equivalent to

$$\begin{aligned} \hat{\psi}_\gamma(t) &= G(t_0, t)^T \hat{\psi}_\gamma(t_0) \\ &\quad + \int_{t_0}^{t} G(\tau, t)^T \left.\frac{\partial Q(\tau, \mathbf{y})}{\partial \mathbf{y}}\right|_{\mathbf{y} = \mathbf{x}(\tau \mid \hat{\mathbf{u}}_\gamma)} d\tau, \qquad \forall t \in [t_0, t_1], \end{aligned} \tag{5.6}$$

where $G(\cdot, s) : [t_0, t_1] \to R^{n \times n}$ is the matrix solution of (3.2) for any $s \in [t_0, t_1]$. Substituting (5.6) into the right-hand side of (5.3), we get

$$\begin{aligned} \hat{\mathbf{u}}_\gamma(t) &= R(t)^{-1} B(t)^T \\ &\quad \times \left[G(t_0, t)^T \hat{\psi}_\gamma(t_0) + \int_{t_0}^{t} G(\tau, t)^T \left.\frac{\partial Q(\tau, \mathbf{y})}{\partial \mathbf{y}}\right|_{\mathbf{y} = \mathbf{x}(\tau \mid \hat{\mathbf{u}}_\gamma)} d\tau \right], \\ &\qquad \forall t \in [t_0, t_1]. \end{aligned} \tag{5.7}$$

Thus,

$$
\begin{aligned}
&\int_{t_0}^{t_1} G(t_1,t)B(t)\hat{\mathbf{u}}_\gamma(t)\,dt \\
&\quad = \left[\int_{t_0}^{t_1} G(t_1,t)B(t)R(t)^{-1}B(t)^T G(t_0,t)^T\,dt\right]\hat{\psi}_\gamma(t_0) \\
&\quad + \int_{t_0}^{t_1}\int_{t_0}^{t} G(t_1,t)B(t)R(t)^{-1}B(t)^T G(\tau,t)^T \left.\frac{\partial Q(\tau,\mathbf{y})}{\partial \mathbf{y}}\right|_{\mathbf{y}=\mathbf{x}(\tau\mid\hat{\mathbf{u}}_\gamma)} d\tau\,dt \\
&\quad = \left[\int_{t_0}^{t_1} G(t_1,t)B(t)R(t)^{-1}B(t)^T G(t_1,t)^T\,dt\right]G(t_0,t_1)^T\hat{\psi}_\gamma(t_0) \\
&\quad + \int_{t_0}^{t_1} G(t_1,\tau)\left[\int_{\tau}^{t_1} G(\tau,t)B(t)R(t)^{-1}B(t)^T G(\tau,t)^T\,dt\right] \\
&\qquad\qquad \times \left.\frac{\partial Q(\tau,\mathbf{y})}{\partial \mathbf{y}}\right|_{\mathbf{y}=\mathbf{x}(\tau\mid\hat{\mathbf{u}}_\gamma)} d\tau.
\end{aligned}
\tag{5.8}
$$

Now, it is clear from (5.8) that

$$
\begin{aligned}
&\left[\int_{t_0}^{t_1} G(t_1,t)B(t)R(t)^{-1}B(t)^T G(t_1,t)^T\,dt\right]G(t_0,t_1)^T\hat{\psi}_\gamma(t_0) \\
&\qquad = \int_{t_0}^{t_1} G(t_1,t)\Big\{B(t)\hat{\mathbf{u}}_\gamma(t) \\
&\qquad\qquad - \left[\int_{t}^{t_1} G(t,\tau)B(\tau)R(\tau)^{-1}B(\tau)^T G(t,\tau)^T\,d\tau\right] \\
&\qquad\qquad \times \left.\frac{\partial Q(t,\mathbf{y})}{\partial \mathbf{y}}\right|_{\mathbf{y}=\mathbf{x}(t\mid\hat{\mathbf{u}}_\gamma)} dt.
\end{aligned}
\tag{5.9}
$$

Note that we can find a constant $C_F > 0$ such that

$$
|M| \le C_F |M|_F, \qquad \forall M \in R^{m\times m}, \tag{5.10}
$$

where

$$
|M| := \max\{|M\mathbf{v}| \mid |\mathbf{v}| = 1, \mathbf{v} \in R^m\}
$$

and

$$
|M|_F := \sqrt{\sum_{i,j=1}^{m} M_{ij}^2},
$$

where M_{ij} is the (i, j)-element of M. Thus, we have

$$\begin{aligned}\forall t \in [t_0, t_1], \mathbf{v} \in R^m, \qquad |R(t)\mathbf{v}|^2 &= \mathbf{v}^T R(t)^2 \mathbf{v} \le |R(t)^2|\,|\mathbf{v}|^2 \\ &\le C_F |R(t)^2|_F |\mathbf{v}|^2 \le C_F |\mathbf{v}|^2 \max_{t_0 \le s \le t_1} |R(s)^2|_F. \end{aligned} \tag{5.11}$$

Taking $\mathbf{v} = R(t)^{-1}\mathbf{w}$ in (5.11), where $\mathbf{w}$ is an arbitrary vector in R^m, we get

$$|R(t)^{-1}\mathbf{w}|^2 \ge \frac{|\mathbf{w}|^2}{C_F \max_{t_0 \le s \le t_1} |R(s)^2|_F}, \qquad \forall \mathbf{u} \in R^m, t \in [t_0, t_1]. \tag{5.12}$$

Relation (5.12) together with (2.3) yields

$$\begin{aligned}\mathbf{w}^T R(t)^{-1}\mathbf{w} &= \left(R(t)^{-1}\mathbf{w}\right)^T R(t) R(t)^{-1}\mathbf{w} \ge \delta_0 |R(t)^{-1}\mathbf{w}|^2 \\ &\ge \frac{\delta_0 |\mathbf{w}|^2}{C_F \max_{t_0 \le s \le t_1} |R(s)^2|_F}, \qquad \forall \mathbf{w} \in R^m, t \in [t_0, t_1]. \end{aligned} \tag{5.13}$$

By (5.13), we obtain

$$\begin{aligned}&\mathbf{y}^T \left[\int_{t_0}^{t_1} G(t_1, t) B(t) R(t)^{-1} B(t)^T G(t_1, t)^T dt \right] \mathbf{y} \\ &\quad = \int_{t_0}^{t_1} \left(B(t)^T G(t_1, t)^T \mathbf{y} \right)^T R(t)^{-1} B(t)^T G(t_1, t)^T \mathbf{y}\, dt \\ &\quad \ge \frac{\delta_0}{C_F \max_{t_0 \le s \le t_1} |R(s)^2|_F} \int_{t_0}^{t_1} |B(t)^T G(t_1, t)^T \mathbf{y}|^2 dt \\ &\quad = \frac{\delta_0}{C_F \max_{t_0 \le s \le t_1} |R(s)^2|_F} \mathbf{y}^T \\ &\qquad \times \left[\int_{t_0}^{t_1} G(t_1, t) B(t) B(t)^T G(t_1, t)^T dt \right] \mathbf{y}, \qquad \forall \mathbf{y} \in R^n. \end{aligned} \tag{5.14}$$

Since (A, B) is controllable,

$$\int_{t_0}^{t_1} G(t_1, t) B(t) B(t)^T G(t_1, t)^T dt$$

is positive definite. Therefore, it is clear from (5.14) that

$$\int_{t_0}^{t_1} G(t_1, t) B(t) R(t)^{-1} B(t)^T G(t_1, t)^T dt$$

is positive definite. Thus, it follows from (5.9) that

$$\hat{\psi}_\gamma(t_0) = G(t_1,t_0)^T \left[\int_{t_0}^{t_1} G(t_1,t) B(t) R(t)^{-1} B(t)^T G(t_1,t)^T \, dt \right]^{-1}$$
$$\times \left\{ \int_{t_0}^{t_1} G(t_1,t) \left\{ B(t) \hat{\mathbf{u}}_\gamma(t) \right.\right.$$
$$- \left[\int_{t}^{t_1} G(t,\tau) B(\tau) R(\tau)^{-1} B(\tau)^T G(t,\tau)^T \, d\tau \right]$$
$$\left.\left. \times \frac{\partial Q(t,\mathbf{y})}{\partial \mathbf{y}} \bigg|_{\mathbf{y}=\mathbf{x}(t \mid \hat{\mathbf{u}}_\gamma)} \right\} dt \right\}. \tag{5.15}$$

On the other hand, by (3.3), we have

$$\begin{aligned} +\infty > \min_{\mathbf{u} \in F} J(\mathbf{u}) &= \min_{\mathbf{u} \in F} J_\gamma(\mathbf{u}) \\ &\geq \min_{\mathbf{u} \in U} J_\gamma(\mathbf{u}) = J_\gamma(\hat{\mathbf{u}}_\gamma) \geq J(\hat{\mathbf{u}}_\gamma) \\ &\geq J(\mathbf{0}) + \int_{t_0}^{t_1} \frac{\delta_0}{2} \left[|\hat{\mathbf{u}}_\gamma(t)| - \frac{C}{\delta_0} \right]^2 dt - \frac{C^2(t_1 - t_0)}{2\delta_0}. \end{aligned} \tag{5.16}$$

Thus, similar to (3.8), we obtain

$$\int_{t_0}^{t_1} |\hat{\mathbf{u}}_\gamma(t)|^2 dt \leq 4 \left\{ \frac{1}{\delta_0} \left[J(\hat{\mathbf{u}}_\gamma) - J(\mathbf{0}) \right] + \left(\frac{C}{\delta_0} \right)^2 (t_1 - t_0) \right\}, \quad \gamma = 1, 2, \ldots \tag{5.17}$$

However, by (5.16), we recall that

$$\min_{\mathbf{u} \in F} J(\mathbf{u}) \geq J(\hat{\mathbf{u}}_\gamma).$$

On this basis, it is clear from (5.17) that

$$\begin{aligned} \int_{t_0}^{t_1} |\hat{\mathbf{u}}_\gamma(t)|^2 dt &\leq 4 \left\{ \frac{1}{\delta_0} \left[\min_{\mathbf{u} \in F} J(\mathbf{u}) - J(\mathbf{0}) \right] + \left(\frac{C}{\delta_0} \right)^2 (t_1 - t_0) \right\} \\ &< +\infty, \qquad \gamma = 1, 2, \ldots \end{aligned} \tag{5.18}$$

In view of (5.18), we can easily show that

$$\sup_{\gamma} \max_{t_0 \leq t \leq t_1} |\mathbf{x}(t \mid \hat{\mathbf{u}}_\gamma)| < +\infty. \tag{5.19}$$

Thus, it follows from (5.15) that

$$\sup_{\gamma} |\hat{\psi}_{\gamma}(t_0)| < +\infty. \tag{5.20}$$

Let $\{\gamma_\mu\}$ be a subsequence of $\{\gamma\}$. Then, by (5.18) and (5.20), there exist a subsequence of $\{\gamma_\mu\}$, which we denote by $\{\gamma_{\mu_\nu}\}$, a $\hat{\mathbf{w}} \in U$, and a $\hat{\psi} \in R^n$ such that

$$\hat{\mathbf{u}}_{\gamma_{\mu_\nu}} \to \hat{\mathbf{w}} \tag{5.21}$$

in the weak topology of $L_2([t_0, t_1], R^m)$, as $\nu \to \infty$ and

$$\hat{\psi}_{\gamma_{\mu_\nu}}(t_0) \to \hat{\psi} \tag{5.22}$$

as $\nu \to \infty$. Now, by (5.21), we have

$$\lim_{\nu \to \infty} \mathbf{x}\left(t \mid \hat{\mathbf{u}}_{\mu_\nu}\right) = \mathbf{x}(t \mid \hat{\mathbf{w}}), \qquad \forall t \in [t_0, t_1]. \tag{5.23}$$

Thus, it follows from (5.6), (5.22)–(5.23), and the Lebesgue dominated convergence theorem that

$$\begin{aligned} \lim_{\nu \to \infty} \hat{\psi}_{\gamma_{\mu_\nu}}(t) = \hat{\psi}(t) &:= G(t_0, t)^T \hat{\psi} \\ &+ \int_{t_0}^{t} G(\tau, t)^T \left. \frac{\partial Q(\tau, \mathbf{y})}{\partial \mathbf{y}} \right|_{\mathbf{y} = \mathbf{x}(\tau \mid \hat{\mathbf{w}})} d\tau, \qquad \forall t \in [t_0, t_1]. \end{aligned} \tag{5.24}$$

Now, by (5.6), it can be verified by virtue of (5.18) and (5.20) that

$$\sup_{\gamma} \max_{t_0 \le t \le t_1} |\hat{\psi}_{\gamma}(t)| < +\infty. \tag{5.25}$$

Thus, by the Lebesgue dominated convergence theorem, (5.3), (5.21), and (5.24), we can show that

$$\lim_{\nu \to \infty} \int_{t_0}^{t_1} |\hat{\mathbf{u}}_{\gamma_{\mu_\nu}}(t) - \hat{\mathbf{w}}(t)|^2 dt = 0 \tag{5.26}$$

and

$$\hat{\mathbf{w}}(\cdot) = R(\cdot)^{-1} B(\cdot)^T \hat{\psi}(\cdot) \in C([t_0, t_1], R^m). \tag{5.27}$$

By (5.26), it follows from the definitions of $\mathbf{x}(\cdot \mid \hat{\mathbf{u}}_{\gamma_{\mu_\nu}})$ and $\mathbf{x}(\cdot \mid \hat{\mathbf{w}})$ that

$$\lim_{\nu \to \infty} \max_{t_0 \le t \le t_1} \left|\mathbf{x}\left(t \mid \hat{\mathbf{u}}_{\gamma_{\mu_\nu}}\right) - \mathbf{x}(t \mid \hat{\mathbf{w}})\right| = 0. \tag{5.28}$$

Therefore, from (5.6), (5.22), and (5.28), we obtain

$$\lim_{\nu\to\infty} \max_{t_0 \le t \le t_1} |\hat{\psi}_{\gamma_{\mu_\nu}}(t) - \hat{\psi}(t)| = 0. \tag{5.29}$$

Combining (5.3), (5.26)–(5.27), and (5.29), we obtain

$$\lim_{\nu\to\infty} \max_{t_0 \le t \le t_1} |\hat{\mathbf{u}}_{\gamma_{\mu_\nu}}(t) - \hat{\mathbf{w}}(t)| = 0. \tag{5.30}$$

Next, we note that

$$J_\gamma(\hat{\mathbf{u}}_\gamma) = \min_{\mathbf{u}\in U} J_\gamma(\mathbf{u}) \le \inf_{\mathbf{u}\in F} J_\gamma(\mathbf{u}) = \min_{\mathbf{u}\in F} J(\mathbf{u}) < +\infty, \qquad \gamma = 1, 2, \ldots \tag{5.31}$$

In view of (5.30) and (5.31), it is clear that

$$\begin{aligned}
\sum_{j=1}^{l} g\Big(E_j(\mathbf{x}(t_1 \mid \hat{\mathbf{w}}))\Big) &= \lim_{\nu\to\infty} \sum_{j=1}^{l} g\Big(E_j\Big(\mathbf{x}\Big(t_1 \mid \hat{\mathbf{u}}_{\mu_\nu}\Big)\Big)\Big) \\
&= \lim_{\nu\to\infty} \frac{1}{\gamma_{\mu_\nu}} \Bigg\{ J_{\gamma_{\mu_\nu}}\Big(\hat{\mathbf{u}}_{\gamma_{\mu_\nu}}\Big) - \Bigg\{ W\Big(\mathbf{x}\Big(t_1 \mid \hat{\mathbf{u}}_{\gamma_{\mu_\nu}}\Big)\Big) \\
&\quad + \int_{t_0}^{t_1} \Big[Q\Big(t, \mathbf{x}\Big(t \mid \hat{\mathbf{u}}_{\gamma_{\mu_\nu}}\Big)\Big) + \frac{1}{2} \hat{\mathbf{u}}_{\gamma_{\mu_\nu}}(t)^T R(t) \hat{\mathbf{u}}_{\gamma_{\mu_\nu}}(t) \Big] dt \Bigg\}\Bigg\} \\
&\le \lim_{\nu\to\infty} \frac{1}{\gamma_{\mu_\nu}} \Bigg\{ \min_{\mathbf{u}\in F} J(\mathbf{u}) - \Bigg\{ W\Big(\mathbf{x}\Big(t_1 \mid \hat{\mathbf{u}}_{\gamma_{\mu_\nu}}\Big)\Big) \\
&\quad \Bigg\{ + \int_{t_0}^{t_1} \Big[Q\Big(t, \mathbf{x}\Big(t \mid \hat{\mathbf{u}}_{\gamma_{\mu_\nu}}\Big)\Big) + \frac{1}{2} \hat{\mathbf{u}}_{\gamma_{\mu_\nu}}(t)^T R(t) \hat{\mathbf{u}}_{\gamma_{\mu_\nu}}(t) \Big] dt \Bigg\}\Bigg\} \\
&= \lim_{\nu\to\infty} \frac{1}{\gamma_{\mu_\nu}} \Bigg\{ \min_{\mathbf{u}\in F} J(\mathbf{u}) - \Big\{ W(\mathbf{x}(t_1 \mid \hat{\mathbf{w}})) \\
&\quad + \int_{t_0}^{t_1} \Big[Q(t, \mathbf{x}(t \mid \hat{\mathbf{w}})) + \frac{1}{2} \hat{\mathbf{w}}(t)^T R(t) \hat{\mathbf{w}}(t) \Big] dt \Big\}\Bigg\} = 0.
\end{aligned} \tag{5.32}$$

Thus,

$$\hat{\mathbf{w}} \in F. \tag{5.33}$$

On the other hand, it follows from (5.31) that

$$\limsup_{\nu\to\infty} J_{\gamma_{\mu_\nu}}\Big(\hat{\mathbf{u}}_{\gamma_{\mu_\nu}}\Big) \le \min_{\mathbf{u}\in F} J(\mathbf{u}). \tag{5.34}$$

By (5.30), we have

$$J(\hat{\mathbf{w}}) = \lim_{\nu\to\infty} J\left(\hat{\mathbf{u}}_{\gamma_{\mu_\nu}}\right) = \liminf_{\nu\to\infty} J\left(\hat{\mathbf{u}}_{\gamma_{\mu_\nu}}\right)$$

$$\leq \liminf_{\nu\to\infty} J_{\gamma_{\mu_\nu}}\left(\hat{\mathbf{u}}_{\gamma_{\mu_\nu}}\right). \tag{5.35}$$

Noting (5.33), and combining (5.34) and (5.35), we obtain

$$J(\hat{\mathbf{w}}) = \min_{\mathbf{u}\in F} J(\mathbf{u}) = \lim_{\nu\to\infty} J_{\gamma_{\mu_\nu}}\left(\hat{\mathbf{u}}_{\gamma_{\mu_\nu}}\right). \tag{5.36}$$

Therefore, by the uniqueness of the optimal control of Problem (P), we have

$$\hat{\mathbf{w}} = \hat{\mathbf{u}}. \tag{5.37}$$

Finally, combining (5.30), (5.36), and (5.37), and noting that $\{\gamma_\mu\}$ is an arbitrary subsequence of $\{\gamma\}$, (5.1) and (5.2) follow readily. This completes the proof.

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